

Squared chromatic and stability numbers without claws or large cliques

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Abstract

Let G be a claw-free graph on n vertices with clique number ω . We prove the following for the square G^2 of G . If $\omega \leq 3$, then its chromatic number satisfies $\chi(G^2) \leq 10$ while its stability number satisfies $\alpha(G^2) \geq n/9$ unless one of its components is a 10-vertex clique. If $\omega \leq 4$, then $\chi(G^2) \leq 22$ and $\alpha(G^2) \geq n/20$. This work is motivated by a conjecture of Erdős and Nešetřil and provides further evidence for a strengthened form of that conjecture.

1 Introduction

Let G be a claw-free graph, that is, a graph without the complete bipartite graph $K_{1,3}$ as an induced subgraph. We consider the square G^2 of G , formed from G by the addition of edges between those pairs of vertices connected by some two-edge path in G . We focus on two parameters of G^2 , its chromatic number $\chi(G^2)$ and its stability number $\alpha(G^2)$. We seek to optimise these with respect to the clique number $\omega(G)$ of G for small $\omega(G)$.

The second author with de Joannis de Verclos and Pastor [8] recently conjectured the following. As the class of claw-free graphs is richer than the class of line graphs (cf. e.g. [2]), this is a significant strengthening of a notorious conjecture of Erdős and Nešetřil (cf. [5]).

Conjecture 1.1 (de Joannis de Verclos, Kang and Pastor [8])

For any claw-free graph G , $\chi(G^2) \leq \frac{1}{4}(5\omega(G)^2 - 2\omega(G) + 1)$ if $\omega(G)$ is odd, and $\chi(G^2) \leq \frac{5}{4}\omega(G)^2$ otherwise.

If true, this would be sharp by the consideration of a suitable blow-up of a five-vertex cycle and taking G to be its line graph. The conjecture of Erdős and Nešetřil is the special case in Conjecture 1.1 of G the line graph of a (simple) graph. To support the more general assertion and at the same time extend a notable result of Molloy and Reed [11], it was proved in [8] that there is an absolute constant $\varepsilon > 0$ such that, provided $\omega(G)$ is sufficiently large, $\chi(G^2) \leq (2 - \varepsilon)\omega(G)^2$ for any claw-free graph G . Moreover, it was proved that Conjecture 1.1 reduces to the case of G the line graph of a multigraph if $\omega(G) \geq 18$.

In this note, our primary goal is to supply additional evidence for Conjecture 1.1 when $\omega(G)$ is small. We affirm it for $\omega(G) = 3$ and come to within 2 of the conjectured value

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when $\omega(G) = 4$. Note that Conjecture 1.1 is trivially true when $\omega(G) \leq 2$.

Theorem 1.2

Let G be a claw-free graph.

- (i) *If $\omega(G) = 3$, then $\chi(G^2) \leq 10$.*
- (ii) *If $\omega(G) = 4$, then $\chi(G^2) \leq 22$.*

Theorem 1.2 extends, in (i), a result independently of Andersen [1] and Horák, Qing and Trotter [7], and, in (ii), a result of Cranston [4]. These earlier results proved¹ the special case in Theorem 1.2 of G the line graph $L(F)$ of some simple graph F .

If a claw-free graph G has n vertices, then Theorem 1.2 implies that the stability number $\alpha(G^2)$ of G^2 (i.e. the number of vertices in a largest stable set of G^2) satisfies $\alpha(G^2) \geq n/10$ if $\omega(G) = 3$ and $\alpha(G^2) \geq n/22$ if $\omega(G) = 4$. This motivates the study of the weaker “dual” search for an optimal lower bound on $\alpha(G^2)$ as a function of n .

Theorem 1.3

Let G be a claw-free graph on n vertices.

- (i) *If $\omega(G) = 3$, then $\alpha(G^2) \geq n/9$ unless a component of G^2 is a 10-vertex clique.*
- (ii) *If $\omega(G) = 4$, then $\alpha(G^2) \geq n/20$.*

Theorem 1.3 extends the results of Joos, Rautenbach and Sasse [10] and Joos and Nguyen [9] who proved the special case of G the line graph $L(F)$ of some simple graph F .

In fact, we have been able to show that, if $\omega(G) \in \{3, 4\}$, then G the line graph of some simple graph is the only case left to prove for Conjecture 1.1. This explains how our results coincide with the best known ones for G a line graph. Our techniques for bounding $\chi(G^2)$ and $\alpha(G^2)$ also apply when $\omega(G) > 4$, but seem to be most effective when $\omega(G)$ is not too large. In particular, our methods and results do not encroach on (and are not encroached upon by) those referred to above applicable for large $\omega(G)$.

It is worth contrasting the work here and in [8] with the extremal study of $\chi(G)$ and $\alpha(G)$ in terms of $\omega(G)$ where in general the situation for claw-free G is markedly different from and more complex than that for G the line graph of some (multi)graph, cf. [3].

Although our reduction for $\omega(G) \in \{3, 4\}$ is qualitatively stronger than that mentioned above for $\omega(G) \geq 18$, this is perhaps only an artefact of the particularities of claw-free graphs with small clique number. Naturally, one could ask if this stronger reduction holds more widely, i.e. if $\omega(G) \in \{5, 6, 7\}$, say, is Conjecture 1.1 “equivalent” to the original conjecture of Erdős and Nešetřil? Failing that, does Conjecture 1.1 reduce to the case of G the line graph of a multigraph for $\omega(G) < 18$? It is conceivable that structural methods such as in [2, 3] may be helpful to resolve these two questions.

It gives insight to notice that the claw-free graphs with clique number at most ω are precisely those graphs each of whose neighbourhoods induces a subgraph with no clique of size $\omega - 1$ and no stable set of size 3. Our results indeed rely on this and related facts. So a good understanding of the graphs that certify small off-diagonal Ramsey numbers should also be useful in the study of the last two mentioned questions.

¹Note the inconsequential overlap when F is a triangle, so the maximum degree $\Delta(F)$ of F is 2 while $\omega(L(F)) = 3$.

Organisation: This note is organised as follows. In the next section, we introduce some basic tools we use. In Section 3, we treat the case $\omega(G) = 3$ and prove Theorems 1.2(i) and 1.3(i). In Section 4, we treat the case $\omega(G) = 4$ and prove Theorems 1.2(ii) and 1.3(ii). In Section 5, we briefly consider the extension of our methods to the case $\omega(G) \geq 5$.

2 Notation and preliminaries

We use standard graph theoretic notation. For instance, if v is the vertex of a graph G , then the neighbourhood of v is denoted $N_G(v)$, and its degree $\deg_G(v)$. We omit the subscripts if this causes no confusion. We frequently make use of the following simple lemmas.

Recall that the Ramsey number $R(k, \ell)$ is the minimum n such that in any graph on n vertices there is guaranteed to be a clique of k vertices or a stable set of ℓ vertices.

Lemma 2.1

Let $G = (V, E)$ be a claw-free graph. For any $v \in V$, the induced subgraph $G[N(v)]$ contains no clique of $\omega(G)$ vertices and no stable set of 3 vertices. In particular, $\deg(v) < R(\omega(G), 3)$.

Proof. If not, then with v there is either a clique of $\omega(G) + 1$ vertices or a claw. \square

Lemma 2.2

Let $G = (V, E)$ be a claw-free graph. For any $v, w \in V$ and $vw \in E$, any two distinct $x, y \in N(w) \setminus (\{v\} \cup N(v))$ are adjacent. In particular, $|N(w) \setminus (\{v\} \cup N(v))| \leq \omega(G) - 1$.

Proof. If not, then v, w, x, y forms a claw. So $\{w\} \cup N(w) \setminus (\{v\} \cup N(v))$ is a clique. \square

It is not required next that $x, y \in N(v)$, but it is the typical context in which it is used.

Lemma 2.3

Let $G = (V, E)$ be a claw-free graph. For any $v \in V$ and $w \in N(v)$, if $N(v) \cap N(w)$ contains two non-adjacent vertices x and y , then for any $z \in N(w) \setminus (\{v\} \cup N(v))$, either $xz \in E$ or $yz \in E$.

Proof. If not, then w, x, y, z forms a claw. \square

3 Clique number three

In this section, we prove Theorems 1.2(i) and 1.3(i). We actually prove the following result.

Theorem 3.1

Let $G = (V, E)$ be a connected claw-free graph with $\omega(G) = 3$. Then one of the following is true:

- (i) G is the icosahedron;
- (ii) G is the line graph $L(F)$ of a graph F of maximum degree 3; or
- (iii) there exists $v \in V$ with $\deg_{G^2}(v) \leq 9$, and, moreover, either $\deg_{G^2}(v) \leq 8$ or there exist y_1, y_2 such that $y_1 \notin N_{G^2}(y_2)$ and $|\{y_1, y_2\} \cup N_{G^2}(y_1) \cup N_{G^2}(y_2)| \leq 18$.

Let us first see how this easily implies Theorems 1.2(i) and 1.3(i).

Proof of Theorem 1.2(i). We prove the result by induction on the number of vertices. The base case trivially holds. We may assume without loss of generality that G is connected. By Theorem 3.1, there are three possibilities. In case (i), $\chi(G^2) \leq 6$ is certified by giving every pair of antipodal points the same colour. In case (ii), the result follows from the strong edge-colouring result of, independently, Andersen [1] and Horák, Qing and Trotter [7]. In case (iii), by induction there is a proper colouring of $G^2 - v$ with 10 colours, and the squared degree of v ensures that there is some available colour for v to produce a proper colouring of G^2 with 10 colours. \square

Proof of Theorem 1.3(i). We prove the result by induction on n . The base case trivially holds. We may assume that G is connected. By Theorem 3.1, there are three possibilities. In case (i), $n = 12$ and any two antipodal points forms a stable set in G^2 . So $\alpha(G^2) \geq n/6$. In case (ii), the result follows from the result of Joos, Rautenbach and Sasse [10]. In case (iii), by induction there is a stable set in $G^2 - (\{v\} \cup N_{G^2}(v))$ with at least $(n-9)/9$ vertices which together with v forms a stable set in G^2 with at least $n/9$ vertices, or there is a stable set in $G^2 - (\{y_1, y_2\} \cup N_{G^2}(y_1) \cup N_{G^2}(y_2))$ with at least $(n-18)/9$ vertices which together with y_1 and y_2 forms a stable set in G^2 with at least $n/9$ vertices. \square

Proof of Theorem 3.1. First note that the maximum degree $\Delta(G)$ of G is at most 5. This follows from Lemma 2.1 and the fact that $R(3, 3) = 6$. Moreover, note that, for any $v \in V$ with $\deg(v) = 5$, $G[N(v)]$ must be a 5-cycle by Lemma 2.1.

For $v \in V$ with $\deg(v) \leq 2$, we have $\deg_{G^2}(v) \leq 2 + 2 \cdot 2 = 6$ by Lemma 2.2.

For $v \in V$ with $\deg(v) = 3$, we have $\deg_{G^2}(v) \leq 3 + 3 \cdot 2 = 9$ by Lemma 2.2.

Let us consider $v \in V$ such that $\deg(v) = 3$ and $\deg_{G^2}(v) = 9$. We write $N(v) = \{w_1, w_2, w_3\}$ and $N(w_i) \setminus (\{v\} \cup N(v)) = \{w_{i,1}, w_{i,2}\}$ for $i \in \{1, 2, 3\}$. Note that by Lemma 2.2 there can be no edge between w_i and $N(w_j) \setminus (\{v\} \cup N(v))$ for distinct $i, j \in \{1, 2, 3\}$, or else there is a clique of four vertices. By symmetry, there are only two cases to consider for $N(v)$: $G[N(v)]$ is the two-edge path $w_1 w_2 w_3$, or $G[N(v)]$ has one edge $w_1 w_2$. In the former case, Lemma 2.3 implies that $w_1 w_{2,i} \in E$ or $w_3 w_{2,i} \in E$ for $i \in \{1, 2\}$, so this case cannot occur. The latter case thus implies good structure for such a v , which we depict in Figure 1. In particular, $G[N(v)]$ is the disjoint union of two cliques if $\deg(v) = 3$ and $\deg_{G^2}(v) = 9$.

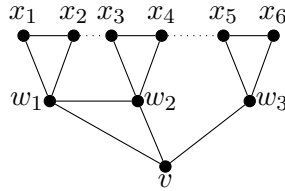


Figure 1: A vertex v with $\deg(v) = 3$ and $\deg_{G^2}(v) = 9$ in the proof of Theorem 3.1.

For $v \in V$ with $\deg(v) = 4$, let us call v *triangular* if it obeys the following structure. There exist distinct $w_1, w_2, w_3, w_4 \in V$ such that $N(v) = \{w_1, w_2, w_3, w_4\}$, $w_1 w_2, w_2 w_3, w_3 w_4 \in E$ and $w_1 w_3, w_2 w_4, w_1 w_4 \notin E$. Moreover, there exist distinct $x_1, x_2, x_3, x_4, x_5 \in V$ such that $N(w_1) \setminus (\{v\} \cup N(v)) = \{x_1, x_2\}$, $N(w_2) \setminus (\{v\} \cup N(v)) = \{x_2, x_3\}$, $N(w_3) \setminus (\{v\} \cup N(v)) = \{x_3, x_4\}$, and $N(w_4) \setminus (\{v\} \cup N(v)) = \{x_4, x_5\}$. See Figure 2.

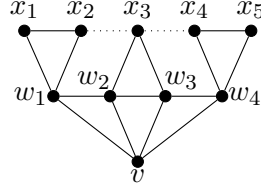


Figure 2: A triangular vertex v .

For $v \in V$ with $\deg(v) \leq 4$, let us call v *good* if neither the subgraph $G[N(v)]$ induced by $N(v)$ is the disjoint union of two cliques nor is v triangular. Let us assume for the moment that G contains no good vertex.

If $\Delta(G) \leq 4$, then every vertex of degree 3 has a neighbourhood that is the disjoint union of a vertex and an edge and every vertex of degree 4 is triangular or has a neighbourhood that is the disjoint union of two edges. Recall that a graph is a line graph if its edges can be partitioned into maximal cliques so that no vertex belongs to more than two such cliques. We can designate the maximal cliques as follows: for $v \in V$ not triangular and a maximal clique C in $N(v)$, designate $v \cup C$ as a maximal clique in the requisite edge partition; for $v \in V$ triangular and $ww' \in G[N(v)]$ such that $|N(\{w, w'\}) \setminus (\{v\} \cup N(v))| > 1$, designate vww' . These designations do not clash for different v . It follows that G is the line graph $L(F)$ of a graph F of maximum degree 3.

If, on the other hand, there exists $v \in V$ with $\deg(v) = 5$, then consider $x \in N(v)$. Since $G[N(v)]$ is a 5-cycle, x has three neighbours y_1, y_2, y_3 that induce a 3-vertex path $y_1y_2y_3$ and $|N(y_2) \setminus (\{x\} \cup N(x))| = 2$. This means x cannot be triangular and $G[N(x)]$ is not the union of two cliques, so x must have degree 5 if no vertex is good. The unique connected graph in which every neighbourhood induces a 5-cycle is the icosahedron.

From now on, let $v \in V$ be a good vertex. Necessarily, $\deg(v) = 4$. To complete the proof, we shall show that $|N(N(v)) \setminus \{v\}| \leq 5$ (which implies $\deg_{G^2}(v) \leq 9$) and moreover that either $|N(N(v)) \setminus \{v\}| \leq 4$ or there exist y_1, y_2 such that $y_1 \notin N_{G^2}(y_2)$ and $|\{y_1, y_2\} \cup N_{G^2}(y_1) \cup N_{G^2}(y_2)| \leq 18$.

Since $G[N(v)]$ has no stable set of three vertices and v is good, $G[N(v)]$ has at least three edges. Moreover, since $G[N(v)]$ has no clique of three vertices, we can write $N(v) = \{w_1, w_2, w_3, w_4\}$ such that $w_1w_2, w_2w_3, w_3w_4 \in E$ and $w_1w_3, w_2w_4 \notin E$. By Lemma 2.2, $|N(w_i) \setminus (\{v\} \cup N(v))| \leq 2$ for $i \in \{1, 2, 3, 4\}$ and $|N(w_i) \cap N(w_j) \setminus (\{v\} \cup N(v))| \leq 1$ for $(i, j) \in \{(1, 2), (2, 3), (3, 4)\}$ (or else there is a clique of four vertices). Moreover, if either $|N(w_2) \setminus (\{v\} \cup N(v))| = 2$ or $|N(w_3) \setminus (\{v\} \cup N(v))| = 2$, then necessarily $|N(w_2) \cap N(w_3) \setminus (\{v\} \cup N(v))| = 1$. By symmetry it remains to consider three cases.

$|N(w_1) \setminus (\{v\} \cup N(v))| \leq 1$: By the above $N(w_2) \cup N(w_3) \setminus (\{v\} \cup N(v))$ has at most three vertices. If $|N(w_2) \cup N(w_3) \setminus (\{v\} \cup N(v))| = 3$, then all but one of them is a neighbour of another vertex in $N(v)$ by Lemma 2.3. So $|N(N(v)) \setminus \{v\}| \leq 1 + 2 + 1 + \lfloor 2/2 \rfloor = 4$. If $|N(w_2) \cup N(w_3) \setminus (\{v\} \cup N(v))| \leq 2$, then each of them is a neighbour of another vertex in $N(v)$. So $|N(N(v)) \setminus \{v\}| \leq 1 + 2 = 3$.

$|N(w_2) \setminus (\{v\} \cup N(v))| \leq 1$: If $|N(w_2) \setminus (\{v\} \cup N(v))| = 0$, then every vertex of $N(w_3) \setminus (\{v\} \cup N(v))$ is a neighbour of another vertex in $N(v)$ by Lemma 2.3, and so $|N(N(v)) \setminus \{v\}| \leq 2 + 2 = 4$. By the above $N(w_2) \cup N(w_3) \setminus (\{v\} \cup N(v))$ has at most two vertices. If $|N(w_2) \cup N(w_3) \setminus (\{v\} \cup N(v))| = 1$, then $|N(N(v)) \setminus \{v\}| \leq 4$ since v is not triangular.

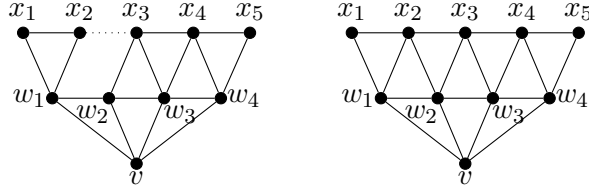


Figure 3: The last two cases in the proof of Theorem 3.1.

So $|N(w_2) \cup N(w_3) \setminus (\{v\} \cup N(v))| = 2$ and by Lemma 2.3 the structure of v must be as follows. There exist $x_1, x_2, x_3, x_4, x_5 \in V$ such that $N(w_1) \setminus (\{v\} \cup N(v)) = \{x_1, x_2\}$, $N(w_2) \setminus (\{v\} \cup N(v)) = \{x_3\}$, $N(w_3) \setminus (\{v\} \cup N(v)) = \{x_3, x_4\}$, and $N(w_4) \setminus (\{v\} \cup N(v)) = \{x_4, x_5\}$. See Figure 3. (Note already that $|N(N(v)) \setminus \{v\}| \leq 5$ for this case.)

In this case, we choose $y_1 := w_1$ and $y_2 := x_4$. Note that $y_1 \notin N_{G^2}(y_2)$. For the estimate on $|\{y_1, y_2\} \cup N_{G^2}(y_1) \cup N_{G^2}(y_2)|$, note that $\{y_1, y_2\} \cup N_{G^2}(y_1) \cup N_{G^2}(y_2)$ includes all of $\{v\} \cup N_{G^2}(v)$ (10 vertices) and some additional elements from $N_{G^2}(y_1)$ and $N_{G^2}(y_2)$. For the extra contribution of $N_{G^2}(y_1)$, we only need to consider $N(x_1) \cup N(x_2) \setminus N(y_1)$. Since $G[N(x_1)]$ contains a vertex of degree 1, namely y_1 , $\deg(x_1) \leq 4$. Similarly $\deg(x_2) \leq 4$. So $|N(x_1) \cup N(x_2) \setminus N(y_1)| \leq 4$. For the extra contribution of $N_{G^2}(y_2)$, let us first note that, since $G[N(x_3)]$ (resp. $G[N(x_5)]$) contains a vertex of degree 1, w_2 (resp. w_4), $\deg(x_3) \leq 4$ (resp. $\deg(x_5) \leq 4$). If $\deg(y_2) = 5$, then there is some common neighbour z of x_3 , y_2 and x_5 . Since $G[N(z)]$ contains a vertex of degree 1, x_3 , $\deg(z) \leq 4$. So $|N_{G^2}(y_2) \setminus (\{v\} \cup N_{G^2}(v))| \leq 4$. We also have $|N_{G^2}(y_2) \setminus (\{v\} \cup N_{G^2}(v))| \leq 4$ if $\deg(y_2) = 4$, since the extra contribution is $|N(x_3) \cup N(x_5) \setminus N(y_2)| \leq 4$. Summing up, we have that $|\{y_1, y_2\} \cup N_{G^2}(y_1) \cup N_{G^2}(y_2)| \leq 10 + 4 + 4 = 18$, as desired.

$|N(w_i) \setminus (\{v\} \cup N(v))| = 2$ for all $i \in \{1, 2, 3, 4\}$: Since there is no clique of four vertices, Lemmas 2.2 and 2.3 imply that $|N(w_i) \cap N(w_j) \setminus (\{v\} \cup N(v))| = 1$ for $(i, j) \in \{(1, 2), (2, 3), (3, 4)\}$. So unless $|N(N(v)) \setminus \{v\}| \leq 4$, the structure of v must be as follows. There exist $x_1, x_2, x_3, x_4, x_5 \in V$ such that $N(w_1) \setminus (\{v\} \cup N(v)) = \{x_1, x_2\}$, $N(w_2) \setminus (\{v\} \cup N(v)) = \{x_2, x_3\}$, $N(w_3) \setminus (\{v\} \cup N(v)) = \{x_3, x_4\}$, and $N(w_4) \setminus (\{v\} \cup N(v)) = \{x_4, x_5\}$. See Figure 3. (Note already that $|N(N(v)) \setminus \{v\}| \leq 5$ for this case.)

In this case, we again choose $y_1 := w_1$ and $y_2 := x_4$ which satisfies $y_1 \notin N_{G^2}(y_2)$. For the estimate on $|\{y_1, y_2\} \cup N_{G^2}(y_1) \cup N_{G^2}(y_2)|$, note that $\{y_1, y_2\} \cup N_{G^2}(y_1) \cup N_{G^2}(y_2)$ includes all of $\{v\} \cup N_{G^2}(v)$ (10 vertices) and some additional elements from $N_{G^2}(y_1)$ and $N_{G^2}(y_2)$. For the extra contribution of $N_{G^2}(y_1)$, we only need to consider $N(x_1) \cup N(x_2) \setminus N(y_1)$. Since $G[N(x_1)]$ contains a vertex of degree 1, namely y_1 , $\deg(x_1) \leq 4$ and $|N(x_1) \setminus N(y_1)| \leq 2$. Since $|N(x_2) \cap N(y_1)| = 4$, $|N(x_2) \setminus N(y_1)| \leq 1$. So $|N(x_1) \cup N(x_2) \setminus N(y_1)| \leq 3$. For the extra contribution of $N_{G^2}(y_2)$, first note that, since $G[N(x_5)]$ contains a vertex of degree 1, w_4 , $\deg(x_5) \leq 4$ and $|N(x_2) \setminus N(y_2)| \leq 2$. Since $|N(x_3) \cap N(y_2)| = 4$, $|N(x_3) \setminus N(y_2)| \leq 1$. If $\deg(y_2) = 4$, then $|N_{G^2}(y_2) \setminus (\{v\} \cup N_{G^2}(v))| \leq 3$. Otherwise, $\deg(y_2) = 5$ and there is some common neighbour z of x_3 , y_2 and x_5 . Since $\deg(z) \leq 5$, $|N_{G^2}(y_2) \setminus (\{v\} \cup N_{G^2}(v))| \leq 4$. Summing up, we have that $|\{y_1, y_2\} \cup N_{G^2}(y_1) \cup N_{G^2}(y_2)| \leq 10 + 3 + 4 = 17$, as desired. \square

4 Clique number four

The proof of Theorem 3.1 suggests the following rougher but more general phenomenon. This follows directly from Lemmas 2.2 and 2.3 together with a double-counting argument.

For $G = (V, E)$ and $v \in V$, we define the following subset of $N(v)$:

$$Z(v) := \{w \in N(v) \mid \exists x, y \in N(v) \text{ such that } xw, wy \in E \text{ and } xy \notin E\}.$$

Lemma 4.1

Let $G = (V, E)$ be a claw-free graph. For any $v \in V$,

$$\begin{aligned} |N(N(v)) \setminus \{v\}| &\leq \sum_{w \in N(v) \setminus Z(v)} |N(w) \setminus (\{v\} \cup N(v))| + \frac{1}{2} \sum_{w \in Z(v)} |N(w) \setminus (\{v\} \cup N(v))| \\ &\leq \left(\deg(v) - \frac{1}{2}|Z(v)| \right) (\omega(G) - 1). \end{aligned}$$

This has the following immediate consequence.

Corollary 4.2

Let $G = (V, E)$ be a claw-free graph. For any $v \in V$ with $\deg(v) \geq 2\omega(G) - 1$,

$$|N(N(v)) \setminus \{v\}| \leq \frac{1}{2} \sum_{w \in N(v)} |N(w) \setminus (\{v\} \cup N(v))| \leq \frac{1}{2} \deg(v) (\omega(G) - 1).$$

Proof. Let $w \in N(v)$ and consider $N_{G[N(v)]}(w)$. By Lemma 2.2, $\deg_{G[N(v)]}(w) \geq \deg(v) - (\omega(G) - 1) - 1 \geq \omega(G) - 1$. Then $N_{G[N(v)]}(w)$ contains a pair of non-adjacent vertices, or else $\{v, w\} \cup N_{G[N(v)]}(w)$ is a clique of $\omega(G) + 1$ vertices. As w was arbitrary, we have just shown that $Z(v) = N(v)$. So the result follows from Lemma 4.1. \square

We now prove the following result. Similar to what we saw if $\omega(G) = 3$, this implies Theorems 1.2(ii) and 1.3(ii) by induction, due to the results in [10] and [9], respectively.

Theorem 4.3

Let $G = (V, E)$ be a connected claw-free graph with $\omega(G) = 4$. Then one of the following is true:

- (i) G is the line graph $L(F)$ of a graph F of maximum degree 4; or
- (ii) there exists $v \in V$ with $\deg_{G^2}(v) \leq 19$.

Proof. First note that the maximum degree $\Delta(G)$ of G is at most 8. This follows from Lemma 2.1 and the fact that $R(3, 3) = 9$.

For $v \in V$ with $\deg(v) \leq 4$, we have $\deg_{G^2}(v) \leq 4 + 4 \cdot 3 = 16$ by Lemma 2.2.

Note that, for $v \in V$ with $\deg(v) = 5$, we have $\deg_{G^2}(v) \leq 5 + 5 \cdot 3 = 20$ by Lemma 2.2, but equality cannot occur here unless $Z(v) = \emptyset$.

For $v \in V$ with $\deg(v) = 5$ and $Z(v) = \emptyset$, $G[N(v)]$ is the disjoint union of cliques, and in particular it must be the disjoint union of an edge and a triangle.

For $v \in V$ with $\deg(v) = 7$, we have $\deg_{G^2}(v) \leq 7 + 21/2 = 17.5$ by Corollary 4.2.

Let $v \in V$ with $\deg(v) = 8$. As we saw in the proof of Corollary 4.2, $Z(v) = N(v)$ and so we already have $\deg_{G^2}(v) \leq 8 + 24/2 = 20$, but we must do one better. Let $w \in N(v)$. By Lemma 2.2, $\deg_{G[N(v)]}(w) \geq \deg(v) - \omega(G) = 4$. Now $N_{G[N(v)]}(w)$ contains no clique or stable set of three vertices, or else G contains a clique of 5 vertices or a claw. We can therefore find four vertices $x_1, x_2, x_3, x_4 \in N_{G[N(v)]}(w)$ such that $x_1x_2, x_3x_4 \notin E$. (There

is at least one non-edge among x_1, x_2, x_3 , say, x_1x_2 . Since G is claw-free at least one of x_1x_3 and x_2x_3 is an edge, say, x_2x_3 . Among x_2, x_3, x_4 , there is at least one non-edge, which together with x_1x_2 or x_1x_3 forms a two-edge matching in the complement, which is what we wanted, after relabelling.) By Lemma 2.3, for every $y \in N(w) \setminus (\{v\} \cup N(v))$, either $x_1y \in E$ or $x_2y \in E$ and $x_3y \in E$ or $x_4y \in E$. We have just shown that every vertex in $N(N(v)) \setminus \{v\}$ has at least three neighbours in $N(v)$. Therefore, $|N(N(v)) \setminus \{v\}| \leq \frac{1}{3} \deg(v)(\omega(G) - 1) = 8$ and $\deg_{G^2}(v) \leq 16$.

Let $v \in V$ with $\deg(v) = 6$. By Lemma 2.2, the minimum degree of $G[N(v)]$ satisfies $\delta(G[N(v)]) \geq \deg(v) - \omega(G) = 2$. Since G contains no clique of 5 vertices, every vertex with degree at least 3 in $G[N(v)]$ must also be in $Z(v)$. So we know there are at most two such vertices, or else by Lemma 4.1 $\deg_{G^2}(v) \leq 6 + \lfloor (6 - 3/2) \cdot 3 \rfloor = 19$. First suppose there is a vertex w with degree 5 in $G[N(v)]$. Since $N_{G[N(v)]}(w)$ contains no clique or stable set of three vertices, it must be that $G[N(v)]$ consists of w adjacent to all vertices of a 5-cycle, in which case all six vertices have degree at least 3 in $G[N(v)]$. Next suppose that there is a vertex w with degree 4 in $G[N(v)]$. Then there exists $w' \in N(v)$ with $ww' \notin E$. As we argued in the last paragraph, there exist $x_1, x_2, x_3, x_4 \in N_{G[N(v)]}(w)$ such that $x_1x_2, x_3x_4 \notin E$. Since G is claw-free, it must be that w' is adjacent to one of x_1 and x_2 and also to one of x_3 and x_4 ; without loss of generality suppose $x_1w', x_3w' \in E$. It follows that x_1, x_3, w are three vertices with degree at least 3 in $G[N(v)]$. So now we have reduced to the case where $2 \leq \delta(G[N(v)]) \leq \Delta(G[N(v)]) \leq 3$ and there are at most two vertices with degree 3 in $G[N(v)]$. Since G is claw-free, there are only two possibilities for the structure of $G[N(v)]$: either it is a disjoint union of two triangles, or it is that graph with the inclusion of exactly one additional edge.

Our case analysis has shown that there must be some $v \in V$ with $\deg_{G^2}(v) \leq 19$, unless for every $v \in V$ the neighbourhood structure satisfies one of the following:

- $G[N(v)]$ is the disjoint union of an edge and a triangle;
- $G[N(v)]$ is the disjoint union of two triangles; or
- $G[N(v)]$ is formed from the disjoint union of two triangles by adding one more edge.

Recall that a graph is a line graph if its edges can be partitioned into maximal cliques so that no vertex belongs to more than two such cliques. We can designate the maximal cliques as follows: for $v \in V$ in one of the first two cases and C a maximal clique in $N(v)$ or for $v \in V$ in the third case and C a *maximum* clique in $N(v)$, designate $v \cup C$ as a maximal clique in the requisite edge partition. These designations do not clash for different v . It follows that G is the line graph $L(F)$ of a graph F of maximum degree 4. \square

5 Clique number at least five (but not too large)

The proof of Theorem 4.3 suggests the following refinement of Lemma 4.1. This might be useful towards reductions to the line graph setting for $\omega(G) \geq 5$ if $\omega(G)$ is not too large.

For $G = (V, E)$ and $v \in V$ and $w \in N(v)$, we define $q(w)$ to be the matching number of the complement of $G[N_{G[N(v)]}(w)]$. Note that $q(w) \geq 1$ if and only if $w \in Z(v)$.

Lemma 5.1

Let $G = (V, E)$ be a claw-free graph. For any $v \in V$,

$$|N(N(v)) \setminus \{v\}| \leq \sum_{w \in N(v)} \frac{|N(w) \setminus (\{v\} \cup N(v))|}{q(w) + 1} \leq (\omega(G) - 1) \sum_{w \in N(v)} \frac{1}{q(w) + 1}.$$

Lemma 5.1 follows directly from Lemmas 2.2 and 2.3 together with a double-counting argument. This yields the following.

Corollary 5.2

Let $G = (V, E)$ be a claw-free graph with $\omega(G) \geq 4$. For any $v \in V$ with $\deg(v) \geq 2\omega(G) - 1$,

$$|N(N(v)) \setminus \{v\}| \leq \frac{\deg(v)(\omega(G) - 1)}{\lceil (\deg(v) + 1)/2 \rceil + 2 - \omega(G)}.$$

Proof. Let $w \in N(v)$. It suffices to establish a suitable lower bound for $q(w)$. By Lemma 2.2, $\deg_{G[N(v)]}(w) \geq \deg(v) - \omega(G) \geq \omega(G) - 1$, and so in any subset of $N_{G[N(v)]}(w)$ with at least $\omega(G) - 1$ vertices there must be at least one non-edge (or else G has a clique of $\omega(G) + 1$ vertices). So we can iteratively extract two vertices from $N_{G[N(v)]}(w)$ that form an edge of the complement of $G[N_{G[N(v)]}(w)]$ until at most $\omega(G) - 2$ vertices remain. It follows that

$$\begin{aligned} q(w) &\geq \left\lceil \frac{1}{2}(\deg_{G[N(v)]}(w) - (\omega(G) - 2)) \right\rceil \geq \left\lceil \frac{1}{2}(\deg(v) - \omega(G) - (\omega(G) - 2)) \right\rceil \\ &= \lceil \deg(v)/2 \rceil + 1 - \omega(G). \end{aligned}$$

If $\deg(v)$ is even, then after we have extracted $\lceil \deg(v)/2 \rceil - \omega(G)$ pairs as above at least $\omega(G)$ vertices remain, call them $x_1, \dots, x_{\omega(G)}$. Among $x_1, \dots, x_{\omega(G)-1}$ there is at least one non-edge, say, $x_1x_2 \notin E$ without loss of generality. Since G is claw-free, it must be that at least one of x_1x_3 and x_2x_3 is an edge, say, $x_2x_3 \in E$ without loss of generality. Since $\omega(G) \geq 4$, among $x_2, \dots, x_{\omega(G)-1}$ there is at least one non-edge, which together with either x_1x_2 or x_1x_3 comprises a two-edge matching in the complement of $G[\{x_1, \dots, x_{\omega(G)-1}\}]$. So indeed we have for any parity of $\deg(v)$ that

$$q(w) \geq \lceil (\deg(v) + 1)/2 \rceil + 1 - \omega(G).$$

As w was arbitrary, the result now follows from Lemma 5.1. \square

An awkward but routine optimisation checks that for $x \in \{2k - 1, 2k, \dots\}$ the expression $x + \frac{x(k-1)}{\lceil (x+1)/2 \rceil + 2 - k}$ is maximised with $x = 2k - 1$ or x odd as large as possible. So by Lemma 2.1 and Corollary 5.2, if v is a vertex of a claw-free graph G with $\deg(v) \geq 2\omega(G) - 1$, then

$$\begin{aligned} \deg_{G^2}(v) &\leq \max \left\{ 2\omega(G) - 1 + (\omega(G) - 1/2)(\omega(G) - 1), \right. \\ &\quad R(\omega(G), 3) - 2 + \frac{(R(\omega(G), 3) - 2)(\omega(G) - 1)}{(R(\omega(G), 3) - 1)/2 + 2 - \omega(G)}, \\ &\quad \left. R(\omega(G), 3) - 1 + \frac{(R(\omega(G), 3) - 1)(\omega(G) - 1)}{R(\omega(G), 3)/2 + 2 - \omega(G)} \right\}. \end{aligned} \tag{1}$$

Moreover, (1) remains valid when we substitute $R(\omega(G), 3)$ with any upper bound. It is known [6] that $R(\omega(G), 3) \leq \binom{\omega(G)+1}{2}$. With this and some routine calculus, (1) implies that $\deg_{G^2}(v) \leq 2\omega(G)(\omega(G) - 1)$ provided $\omega(G) \geq 3$, and $\deg_{G^2}(v) \leq \frac{1}{4}(5\omega(G)^2 - 2\omega(G) + 1) - 1$

provided $\omega(G) \geq 5$. The former yields that $\chi(G^2) \leq 2\omega(G)(\omega(G) - 1) + 1$ for any claw-free G (since those v with $\deg(v) \leq 2\omega(G) - 2$ have $\deg_{G^2}(v) \leq 2\omega(G)(\omega(G) - 1)$ by Lemma 2.2) — this “trivial” bound on $\chi(G^2)$ was proved by a different method in [8]. The latter implies that Conjecture 1.1 holds if it holds for all G with $\Delta(G) \leq 2\omega(G) - 2$, a reduction that is of interest for $5 \leq \omega(G) < 18$.

For $\omega(G) = 5$, $\deg_{G^2}(v) \leq 30$ if $\deg(v) \leq 6$ by Lemma 2.2 and Conjecture 1.1 posits that $\chi(G^2) \leq 29$, so to get within 2 of the conjecture, it suffices to consider when $7 \leq \delta(G) \leq \Delta(G) \leq 8$. Similarly, to get within 4 of the conjecture of 45 for $\omega(G) = 6$, it suffices to consider when $9 \leq \delta(G) \leq \Delta(G) \leq 10$, while, to get within 6 of the conjecture of 58 for $\omega(G) = 7$, it suffices to consider when $10 \leq \delta(G) \leq \Delta(G) \leq 12$.

It is then natural to wonder if Conjecture 1.1 for $\omega(G) \in \{5, 6, 7\}$ hinges on the original conjecture of Erdős and Nešetřil for the corresponding cases. For the original cases, however, there has been little progress: respectively, the trivial bound yields 41, 61, 85, Cranston [4] speculates that 37, 56, 79 are within reach, and the conjectured values are 29, 45, 58.

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